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Analytic Treatment of the Relativistic Motion of Charged Particles in Electric and Magnetic Fields.

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Summary. — Using the Boris mover, or other standard methods, the error pile-up in the computation of the relativistic orbit of charged particles in electric and magnetic fields becomes quickly excessive, if one wants to keep reasonably limited the number of points employed to build the particle orbit. An analytical solution becomes, therefore, desirable and its construction is the subject of the present work.

PACS. 41.70. – Particles in electromagnetic fields: classical aspects (including synchrotron radiation).

1. – Introduction.

The computer calculation of charged-particle orbits in the presence of constant electric and magnetic fields cannot be exploited with a simple «forward integration» of the dynamic equation of motion, because an excessive error is piled up, and, moreover, the procedure is not symmetrical with regard to time reversal.

Particle orbits must be computed, instead, using a «leapfrog» scheme, where the finite-difference form of derivatives is kept symmetrical with respect to time reversal [1].

The acceleration of a particle by a space-dependent force can easily be dealt with symmetrically by using «central differences» in the form

$$\frac{\boldsymbol{X}_{n+1} - 2\boldsymbol{X}_n + \boldsymbol{X}_{n-1}}{\delta t^2} \simeq \boldsymbol{F}_n/m. \tag{1}$$

(1) O. BUNEMAN: *J. Comput. Phys.* **1**, 517 (1967); **12**, 124 (1973).

The incorporation of a Lorentz force, however, raises the difficulty of representing the velocity symmetrically in the cross-product $V \times B$. Electrodynamics, indeed, is symmetric with regard to the time reversal, provided one changes the sign of the magnetic field along with the sign of time.

Representing the velocity in the form $V = (X_{n+1} - X_{n-1})/2\,\delta t$, an explicit solution for X_{n+1} can be obtained from the Lorentz equation in the finite-difference form

$$\frac{X_{n+1} - 2X_n + X_{n-1}}{\delta t^2} = e\left\{E(X_n) + \frac{X_{n+1} - X_{n-1}}{2c \cdot \delta t} \times B(X_n)\right\}\Big/ m . \tag{2}$$

The presence of X_n is sufficient to prevent the build-up of odd-even discrepancies.

Equation (2) can be rewritten utilizing the velocities at half-integral time levels

$$[V_{n+\frac{1}{2}} - V_{n-\frac{1}{2}}]/\delta t = e\,\{E_n + [V_{n+\frac{1}{2}} + V_{n-\frac{1}{2}}] \times B_n/2c\}/m . \tag{3}$$

The vector equation (3) can be solved for $V_{n+\frac{1}{2}}$ as a system of three simultaneous scalar equations.

The solution given by the so-called «Boris mover» ([2]), using several steps and a complete separation of electric and magnetic forces, is better and simpler: the separation of parallel and perpendicular components of motion is not needed (as in the similar Buneman method ([3])) and the relativistic generalization is straightforward ([4]).

Recalling that the angle of rotation between consecutive velocities, normal to the magnetic field B, is close to $\omega\,\delta t$ (with $\omega = eB/mc$), it can be shown that the «Boris mover» produces, for the time step δt, a rotation angle equal to

$$\omega\,\delta t[1 - (\omega\,\delta t)^2/12 + \ldots] ,$$

where the error is less than the second term of the series expansion (fig. 1).

This is, however, a finite-difference method, where, following the particle along its orbit, the error piling up becomes excessive when $\omega\delta t$ is larger than one radian.

Moreover, the condition with high magnetic field must be considered, where

([2]) J. P. Boris: *Relativistic Plasma Simulation*, in *Proceedings of the Fourth Conference on Numerical Simulation of Plasmas*; edited by J. Boris and R. Shanny (NRL, Washington, D. C., 1970) p. 3.

([3]) O. Buneman, C. W. Barnes, J. C. Green and D. E. Nielsen: *J. Comput. Phys.* 38, 1 (1980).

([4]) C. K. Birdsall, A. Bruce and I. Langdon: *Plasma Physics Via Computer Simulation* (McGraw-Hill New York, N. Y., 1985).

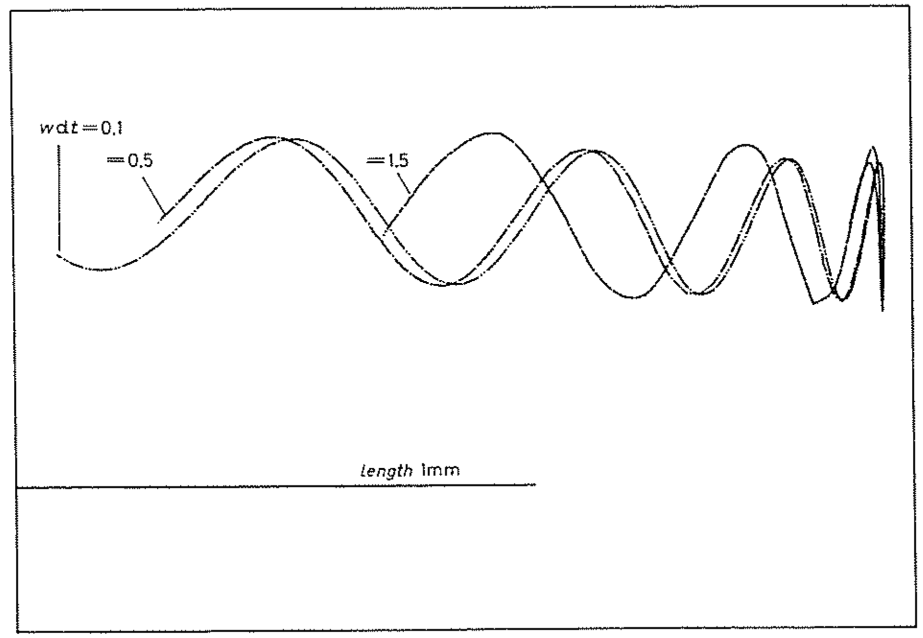

Fig. 1. – Electron orbit calculated with the Boris method. A very large number of points is necessary to build up the orbit inside a cell. The length indicated on the right is 1 mm. The initial value of particle's β is 0.55. After $2.4 \cdot 10^{-11}$ s (time of the laboratory frame) β becomes 0.64. The electric field has intensity $4.9 \cdot 10^5$ V/cm. The magnetic field is $7.9 \cdot 10^4$ G. The orbit is projected in the (Z, X)-plane of the reference frame L.

the rotation motion of the particles gives a long path even when the displacement of the guiding centre is very short. In the Boris method one must calculate a sequence of points along the particle orbit, so that the orbit build-up must be performed inside each cell with a very high number of points.

An analytical solution becomes, therefore, desirable; its construction is the subject of the present work.

The analytical solution of motion equations for relativistic particles moving in electric and magnetic fields is possible only in two cases: when the electric field and the magnetic field are parallel, and when the fields are mutually orthogonal and, in Gaussian units (adopted in this paper), with the same intensity.

An analytical solution of the relativistic motion equations of charged particles is made all the more interesting by the present need to use their orbits in parametric form for the application in plasma simulation codes.

During a numerical plasma simulation, one alternates the fields calculation, starting from electric charge and current distribution, and the solution of the particle motion equations, starting from electric- and magnetic-field distribution and from the boundary conditions, in order to «update» the particle positions and velocities.

Total field changes in space and time are obviously due to the particle kinematics (*i.e.* to their position and velocity).

During each time step, the particle motion is built up assuming for the fields the values calculated in the former step, and the extent of the time interval δt and that of the spatial cell are dictated by the requirement that changes of $\boldsymbol{E}$ and $\boldsymbol{B}$, as seen by each particle, can be ignored. Therefore, the analytical solutions must be contained in cells, where $\boldsymbol{E}$ and $\boldsymbol{B}$ can be considered constant.

2. – Reference frames L and K.

In order to obtain the analytical solutions of the relativistic motion equations it is necessary to find, starting from the laboratory frame L, a reference frame K', where the transformed fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ are parallel, and a final frame K'' obtained rotating K' in such a way that $\boldsymbol{E}'$ (and $\boldsymbol{B}'$) are along the $\hat{\boldsymbol{Z}}''$ axis.

It is preliminarly necessary to define a frame of reference K, where the vector $\boldsymbol{B}$ has only the component B_{3K} with respect to the $\hat{\boldsymbol{Z}}_K$ axis, and the vector $\boldsymbol{E}$ lies in the ($\hat{\boldsymbol{Y}}_K$, $\hat{\boldsymbol{Z}}_K$) plane (see table I).

TABLE I.

Reference frame	Field vector	Components	Coordinates
L	$\boldsymbol{E}, \boldsymbol{B}$	$E_1, E_2, E_3, B_1, B_2, B_3$	X_L, Y_L, Z_L
K	$\boldsymbol{E}, \boldsymbol{B} \equiv \boldsymbol{E}_K, \boldsymbol{B}_K$	E_{2K}, E_{3K}, B_{3K}	X_K, Y_K, Z_K
K'	$\boldsymbol{E}', \boldsymbol{B}'$	E_2', E_3', B_2', B_3'	X', Y', Z'
K''	$\boldsymbol{E}', \boldsymbol{B}' \equiv \boldsymbol{E}'', \boldsymbol{B}''$	E_3'', B_3''	X'', Y'', Z''

If $E_1, E_2, E_3, B_1, B_2, B_3$ are the components of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ in the laboratory frame L, assuming as positive direction of the $\hat{\boldsymbol{Z}}_K$ axis the direction of the magnetic field $\boldsymbol{B}$, the components of the versor $\hat{\boldsymbol{Z}}_K$ are

(4) $$C(3, i) = B_i/|\boldsymbol{B}| \qquad (i = 1, 2, 3).$$

The $\hat{\boldsymbol{X}}_K$ axis is parallel to the vector

$$\boldsymbol{I} = (\boldsymbol{E}/E) \times \hat{\boldsymbol{Z}}_K, \qquad \text{where } E = |\boldsymbol{E}|.$$

If we put $CE(i) = E_i/E$, the components of the vector $\boldsymbol{I}$ are given by

(5) $$\begin{cases} I1 = CE(2)\cdot C(3,3) - CE(3)\cdot C(3,2), \\ I2 = CE(3)\cdot C(3,1) - CE(1)\cdot C(3,3), \\ I3 = CE(1)\cdot C(3,2) - CE(2)\cdot C(3,1). \end{cases}$$

The unit vector of the $\hat{\boldsymbol{X}}_K$ axis is, therefore, $\hat{\boldsymbol{X}}_K = \boldsymbol{I}/|\boldsymbol{I}|$, whose components are

(6) $$C(1, i) = I_i/|\boldsymbol{I}| \qquad (i = 1, 2, 3).$$

The unit vector $\hat{\boldsymbol{Y}}_K = \hat{\boldsymbol{Z}}_K \times \hat{\boldsymbol{X}}_K$ has the components

(7) $$\begin{cases} C(2,1) = C(3,2)\cdot C(1,3) - C(3,3)\cdot C(1,2), \\ C(2,2) = C(3,3)\cdot C(1,1) - C(3,1)\cdot C(1,3), \\ C(2,3) = C(3,1)\cdot C(1,2) - C(3,2)\cdot C(1,1). \end{cases}$$

The system $\hat{\boldsymbol{X}}_K$, $\hat{\boldsymbol{Y}}_K$, $\hat{\boldsymbol{Z}}_K$ is defined in the frame of reference $\hat{\boldsymbol{X}}_L$, $\hat{\boldsymbol{Y}}_L$, $\hat{\boldsymbol{Z}}_L$ by the direction cosines $C(i,j)$. The coordinates transformation from L to K is given by

(8) $$\begin{cases} X_K = X_L\cdot C(1,1) + Y_L\cdot C(1,2) + Z_L\cdot C(1,3), \\ Y_K = X_L\cdot C(2,1) + Y_L\cdot C(2,2) + Z_L\cdot C(2,3), \\ Z_K = X_L\cdot C(3,1) + Y_L\cdot C(3,2) + Z_L\cdot C(3,3). \end{cases}$$

Now that the reference frame K is defined, it is possible to calculate in this system the $\boldsymbol{E}$ and $\boldsymbol{B}$ components, initially assigned in the system L. We get

(9) $$\begin{cases} E_{2K} = \sum_{i=1}^{3} E_i\cdot C(2,i), \quad E_{3K} = \sum_{i=1}^{3} E_i\cdot C(3,i), \quad B_{3K} = \sum_{i=1}^{3} B_i\cdot C(3,i), \\ E_{1K} = B_{1K} = B_{2K} = 0. \end{cases}$$

3. – Reference frames K' and K''.

The origins of the two frames K and K' are assumed to coincide at the time $t = 0$ (fig. 2).

K' is assumed to move with respect to the frame K with a velocity $\boldsymbol{V}$ in the direction of the $\hat{\boldsymbol{X}}_K$ axis (collinear to $\hat{\boldsymbol{X}}'$).

Putting

(10) $$\beta_K = V/c, \qquad \gamma_K = (1-\beta_K^2)^{-\frac{1}{2}}$$

the fields must satisfy the Lorentz transformations

(11) $$E_1' = 0, \quad E_2' = \gamma(E_{2K} - \beta_K B_{3K}), \quad E_3' = \gamma_K E_{3K},$$

(12) $$B_1' = 0, \quad B_2' = \gamma_K\beta_K E_{3K}, \quad B_3' = \gamma_K(B_{3K} - \beta_K E_{2K}).$$

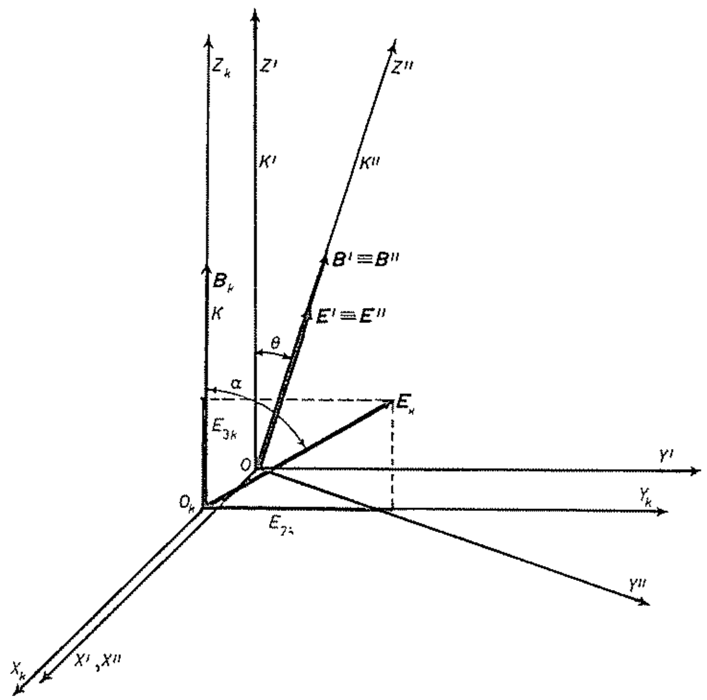

Fig. 2. – Frames K, K' and K''. K' moves with velocity $\boldsymbol{V}$ in the direction of the $\hat{X}_K$ axis ($\hat{X}'$ is collinear to $\hat{X}_K$).

$\boldsymbol{E}'$ and $\boldsymbol{B}'$ become parallel in the frame K' if

$$B'_2/B'_3 = E'_2/E'_3 ,$$

so that

$$E_{3K}\beta_K/(B_{3K} - E_{2K}\beta_K) = (E_{2K} - \beta_K B_{3K})/E_{3K} \tag{13}$$

from which, if $E_{3K} \neq 0$, β_K is obtained as a root of the equation

$$\beta_K^2 - \beta_K(E_{2K}^2 + E_{3K}^2 + B_{3K}^2)/E_{2K}\cdot B_{3K} + 1 = 0 . \tag{14}$$

The solution is given by

$$\beta_K = -b/2 \pm \sqrt{b^2/4 - 1} , \tag{15}$$

where $b = -(E_{2K}^2 + E_{3K}^2 + B_{3K}^2)/E_{2K}B_{3K}$. β_K is real for $b > 2$, which is always found to be verified. The correct solution needed by our transformation is that with the minus sign, such that $\beta_K \to 0$ for $E_{2K} \to 0$.

By construction, the only field component that (in the frame K) can assume a negative sign is E_{3K}.

For $E_{3K} < 0$ B'_2 and E'_3 shall also be < 0, while E'_2 and B'_3 shall always be > 0 (fig. 3*a*)).

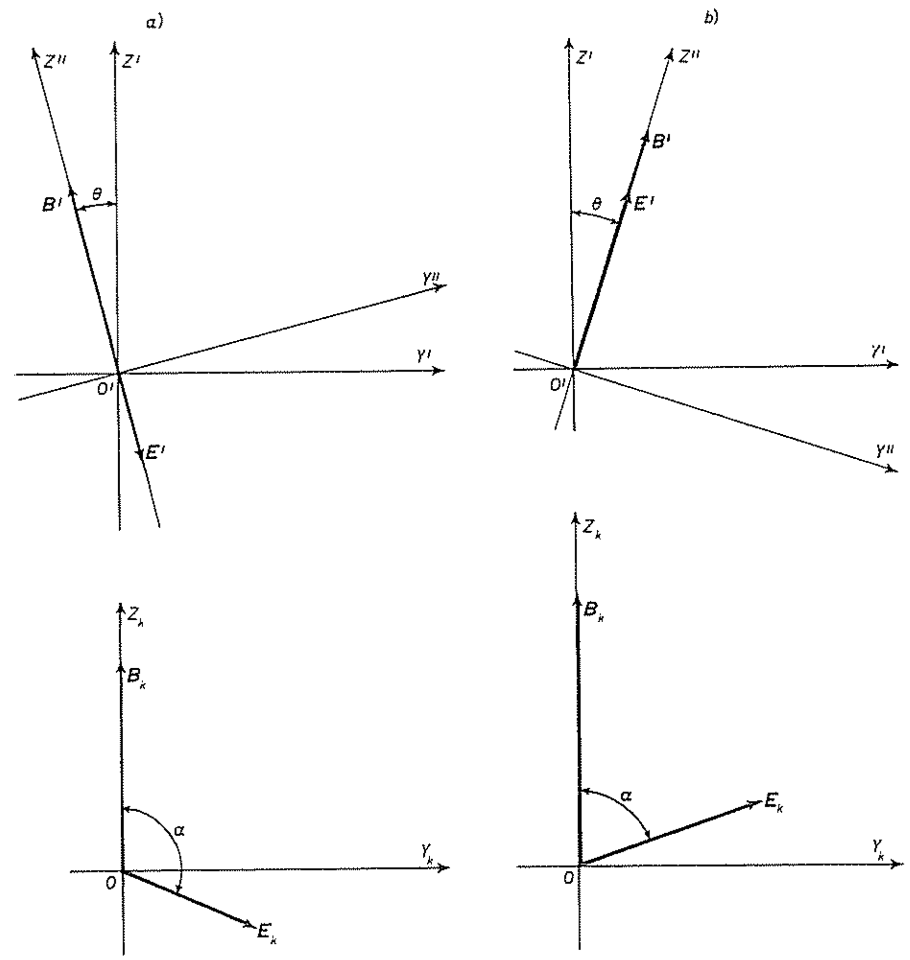

Fig. 3. – Construction of the frame K'' from K' by rotation about the X' axis of the angle θ. *a*) For $E_{3K} < 0$ ($\alpha > \pi/2$), where α is the angle between $\boldsymbol{B}$ and $\boldsymbol{E}$ in the system K. *b*) For $E_{3K} > 0$ ($\alpha < \pi/2$).

For $E_{3K} > 0$ all the components of the two transformed vectors $\boldsymbol{E}'$ and $\boldsymbol{B}'$ shall be > 0 (fig. 3*b*)).

For $E_{3K} \to 0$ the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ approach to be orthogonal.

Three cases are possible in this condition: $E_{2K} \lesseqgtr B_{3K}$.

a) For $E_{2K} \neq B_{3K}$ ($E_{3K} \to 0$) we obtain from solution (15), recalling that β_K must be less than unity,

$$(16) \qquad \beta_K = (E^2_{2K} + B^2_{3K})/(2 \cdot E_{2K} \cdot B_{3K}) - [(E^2_{2K} + B^2_{3K})^2/(4 \cdot E^2_{2K} \cdot B^2_{3K}) - 1]^{\frac{1}{2}} =$$
$$= [E^2_{2K} + B^2_{3K} - (\pm E^2_{2K} \mp B^2_{3K})]/(2E_{2K} \cdot B_{3K})$$

so that

$$(17) \qquad \beta_K = B_{3K}/E_{2K} \qquad \text{for } E_{2K} > B_{3K},$$

$$(18) \qquad \beta_K = E_{3K}/B_{2K} \qquad \text{for } E_{2K} < B_{3K}.$$

These values of β_K are the same obtained when, for orthogonal $\boldsymbol{E}$ and $\boldsymbol{B}$, we choose a reference frame moving in such a way that one of the two fields is cancelled by the relativistic transformation required by the assigned movement [5].

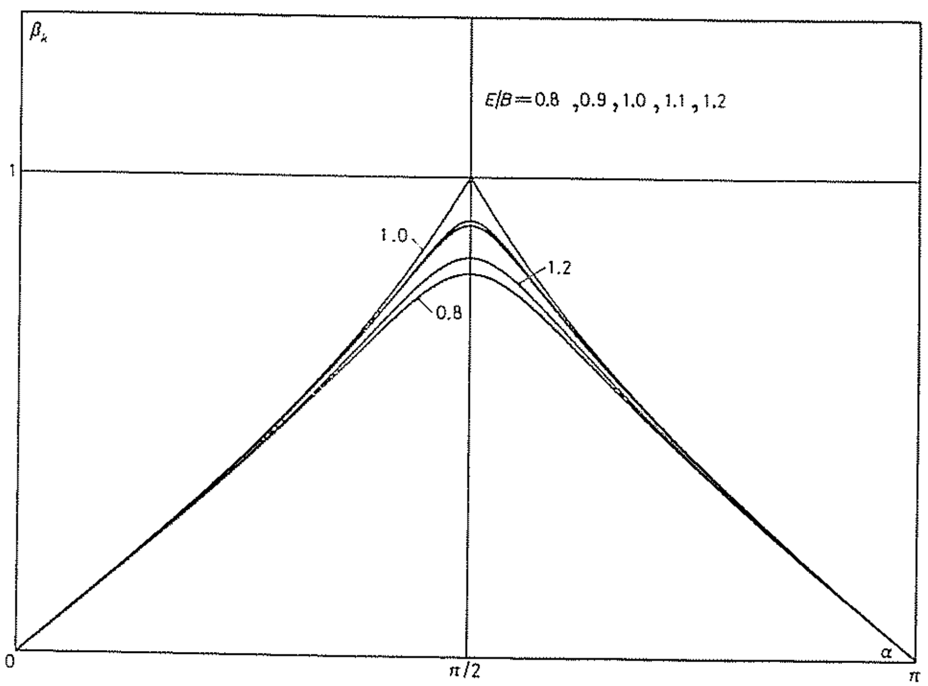

Fig. 4. – $\beta_K = V/c$ *vs.* α. For $E_{2K} = B_{3K}$ and $E_{3K} = 0$ (Gaussian units) we have $\beta_K = 1$, *i.e.* a physically impossible limiting case.

([5]) J. D. JACKSON: «*Classical Electrodynamics*», (Wiley, New York, N. Y., 1962).

b) When $E_{2K} = B_{3K}$ ($E_{3K} \to 0$) transformations (17) and (18) cannot be fulfilled because they would imply $\beta_K = 1$.

In this case, however, another analytic solution exists [6], which can be expressed in parametric form, and shall be treated in sect. 4, case *a*).

From β_K we can obtain the angle θ, which is the angle between the two transformed vectors ($\boldsymbol{E}'$ and $\boldsymbol{B}'$) and the vector $\boldsymbol{B}$ (see fig. 2).

The angle θ is given by

$$(19) \qquad \theta = \mathrm{tg}^{-1}(B'_2/B'_3) = \mathrm{tg}^{-1}[\beta_K E_{3K}/(B_{3K} - \beta_K E_{2K})]$$

or by

$$(20) \qquad \theta = \mathrm{tg}^{-1}(E'_2/E'_3) = \mathrm{tg}^{-1}[(E_{2K} - \beta_K B_{3K})/E_{3K}].$$

When, in the frame K, $B = E$ we have

$$(21) \qquad \mathrm{tg}\,\theta = [\mathrm{sign}\,(E_{3K}) \cdot E - E_{3K}]/E_{2K},$$

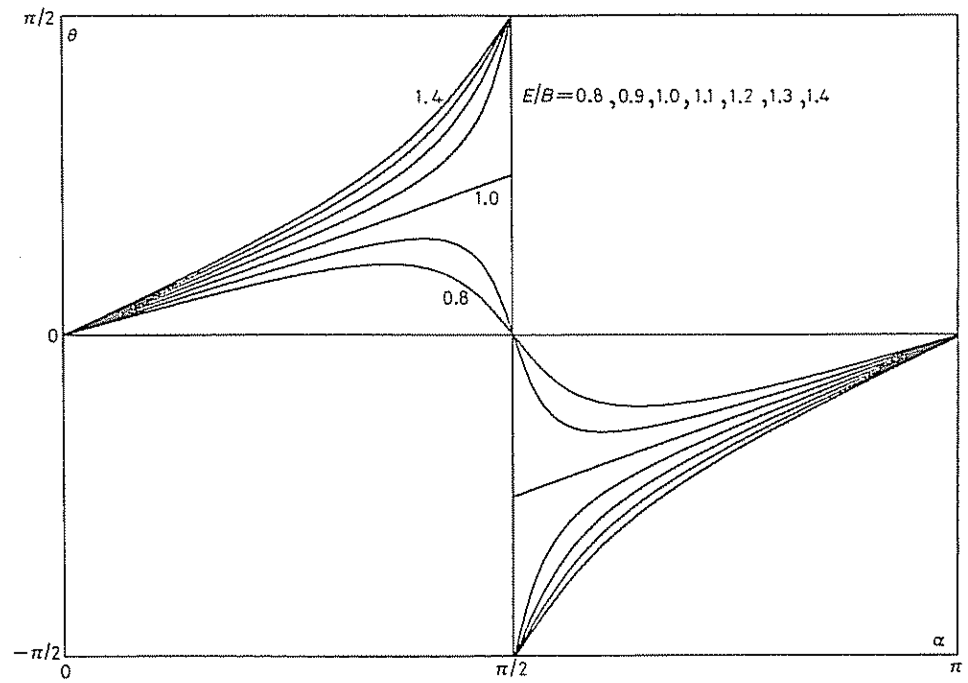

Fig. 5. – Plot of the angle θ (the $\hat{Z}'$ rotation angle to obtain the $\hat{Z}''$-axis) *vs.* α, for different values of the parameter E/B.

([6]) L. LANDAU and E. LIFCHITZ: *Theorie du Champ* (MIR, Moscow, 1966).

from which, being α the angle between $\boldsymbol{B}$ and $\boldsymbol{E}$, we obtain

$$\theta=\alpha/2 \quad \text{for } 0<\alpha<\pi/2\,, \qquad \theta=-(\pi-\alpha)/2 \quad \text{for } \pi/2<\alpha<\pi$$

(see fig. 4 and 5).

For $E_{3K}<0$ ($E_{2K}\neq 0$) being B_2' and $E_3'<0$, θ shall be <0.

For $E_{3K}>0$ the angle θ is >0.

The last possible case is when $E_{2K}=0$, so that the fields are already parallel in the systems K and L. In this case the system K' coincides with K ($\beta_K=0$) (see fig. 6-11 for a complete description of all the parameters as a function of the angle α).

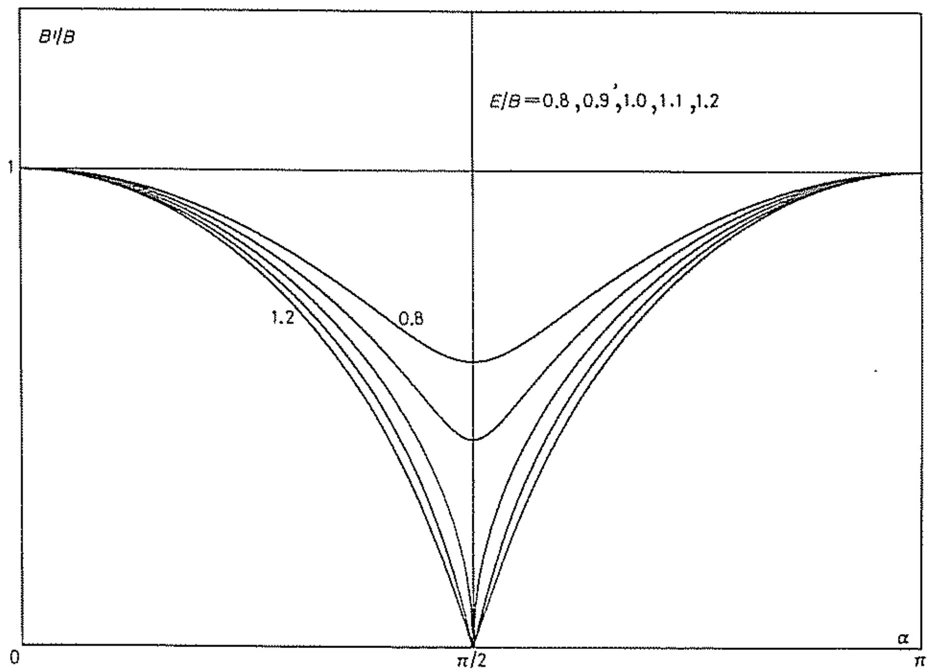

Fig. 6. – B'/B *vs.* α for different values of the ratio E/B.

We introduce now a fourth frame of reference K'' (obtained from K' rotating it about the X' axis by the angle θ, so that the $\hat{\boldsymbol{Z}}''$ axis is directed as $\boldsymbol{E}'$ and $\boldsymbol{B}'$), where the particle orbit can be calculated by the known analytic solutions (see fig. 2).

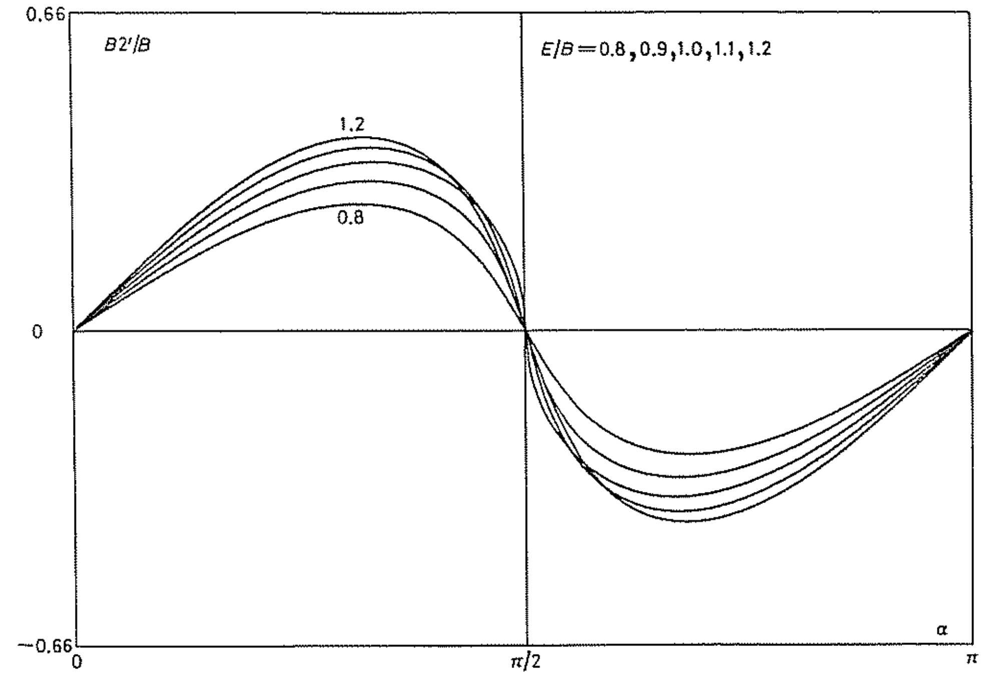

Fig. 7. – B_2'/B *vs.* α.

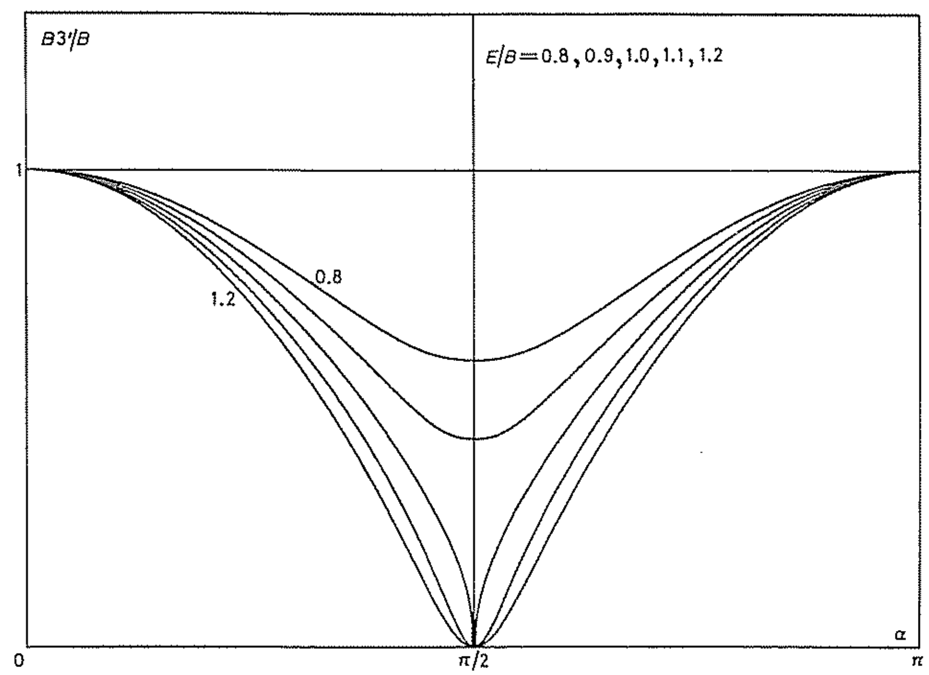

Fig. 8. – B_3'/B *vs.* α.

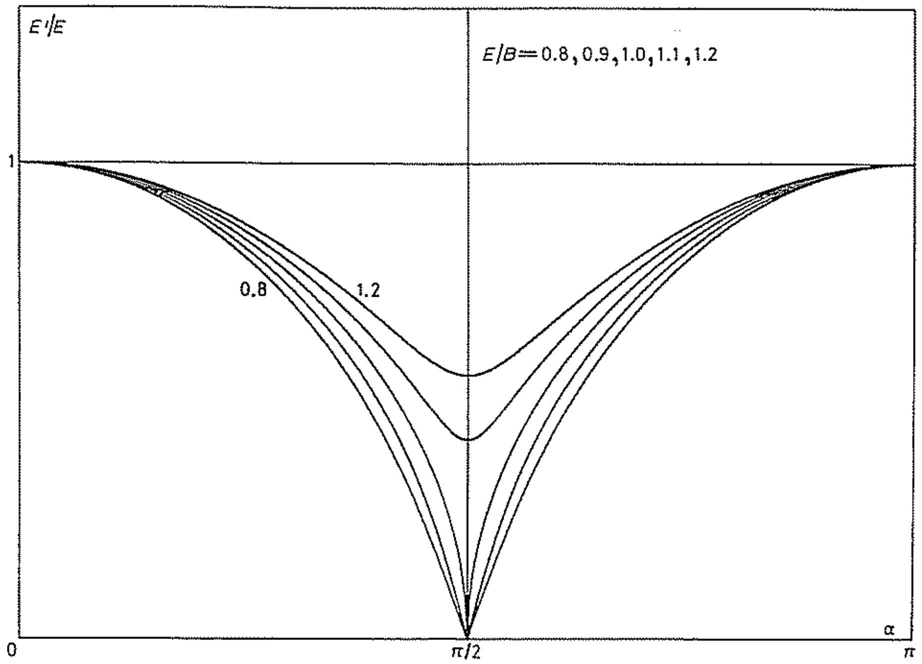

Fig. 9. – E'/E *vs.* α.

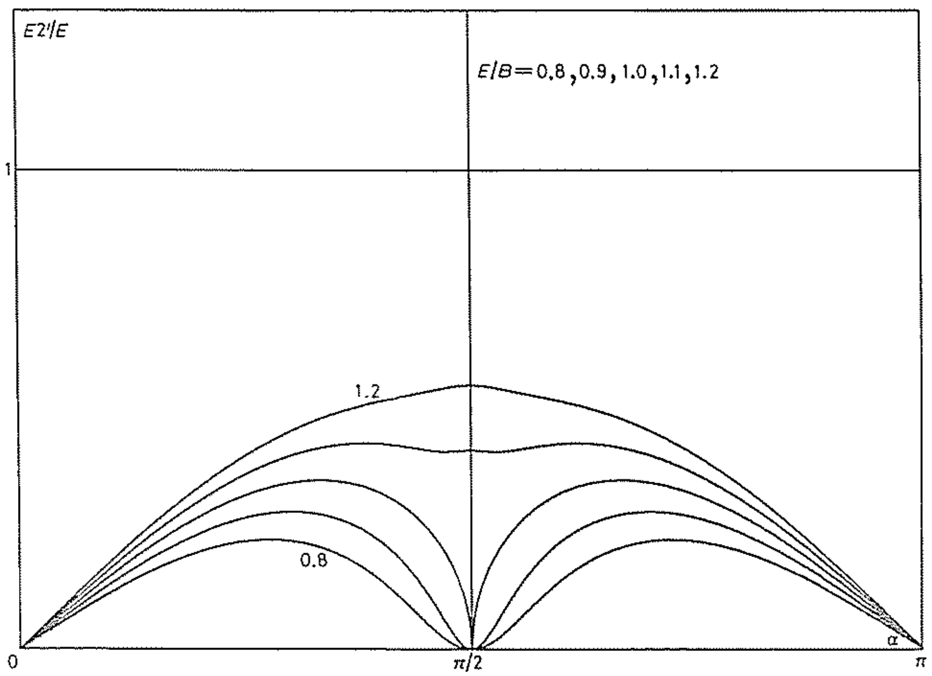

Fig. 10. – E_2'/E *vs.* α.

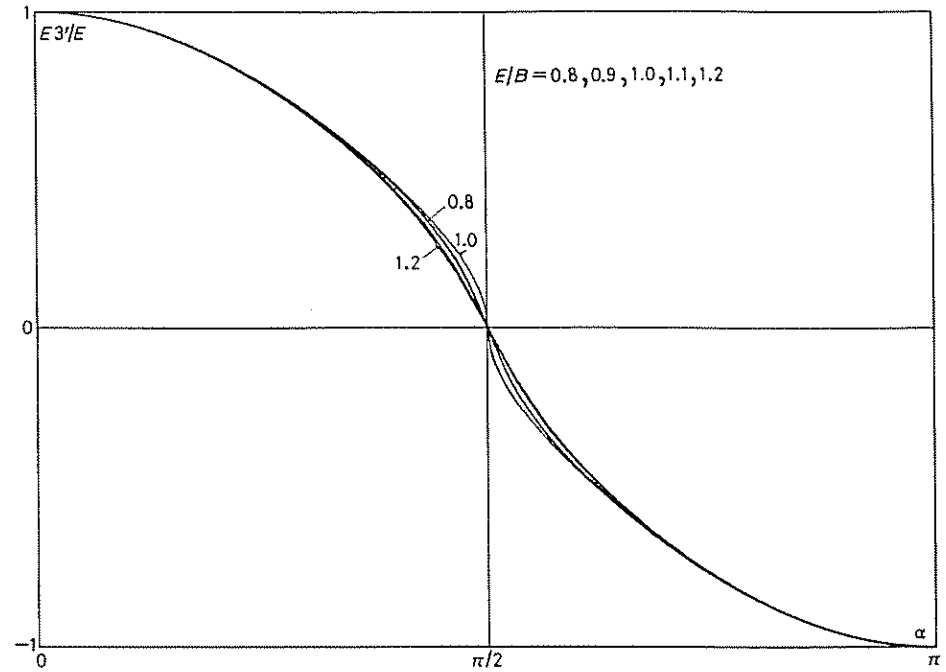

Fig. 11. – E_3'/E *vs.* α. The single components of the transformed vectors $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in the frame of reference K' are very different for $\boldsymbol{E}'$ and $\boldsymbol{B}'$, but the ratios of the two vector amplitudes: $|\boldsymbol{E}'|/|\boldsymbol{E}|$, $|\boldsymbol{B}'|/|\boldsymbol{B}|$ (fig. 6 and 9) are quite similar.

4. – Analytic solutions.

In the system L the initial momentum $\boldsymbol{P}_0$ is assigned by its components $P_0(i)$, together with the initial coordinates $X_0(i)$.

The transformation from the reference frame L to the frame K is defined by

$$P_{0K}(i) = \sum_{j=1}^{3} P_0(j)\cdot C(i,j)\,, \tag{22}$$

$$X_{0K}(i) = \sum_{j=1}^{3} X_0(j)\cdot C(i,j)\,. \tag{23}$$

The relationship between the frame of reference K and the frame K' is expressed by the relativistic transformations

$$\begin{cases} X' = \gamma_K[X_K - \beta_K t\cdot c]\,, \\ Y' = Y_K\,, \\ Z' = Z_K\,, \\ t' = \gamma_K[t - \beta_K X_K/c]\,. \end{cases} \tag{24}$$

It is assumed that for $t=0$ the particle coordinates are

$$X_K = Y_K = Z_K = 0\,.$$

The particle total energy in the system K is

$$Et = m_0 c^2 \gamma\,, \qquad \text{where } \gamma = [1 + P_K^2/m_0^2 c^2] \tag{25}$$

being $P_K^2 = P_K(1)^2 + P_K(2)^2 + P_K(3)^2$ in the system K ($\boldsymbol{P}_K = \boldsymbol{P}_L$).

Going to the system K', the total energy is transformed as

$$Et' = \gamma_K[Et - \beta_K \cdot P_0(1) \cdot c]\,. \tag{26}$$

The momentum transformation leaves its components unchanged with respect to the Y_K and Z_K axes, while the component with respect to the X_K axis is transformed by the equation

$$P'(1) = \gamma_K[P_K(1) - Et\beta_K/c]\,. \tag{27}$$

a) In the system K for $E_{2K} = B_{3K}$ and $E_{3K} = 0$ the solution is given in a parametric form (6).

The parameter, monotonous in time because the electric-field component E_{2K} is always directed as the Y_K axis, is the particle momentum $RP_K(2)$ along Y_K.

From the equations of the relativistic dynamics in this very peculiar case, we have, in the reference frame K,

$$\begin{cases} \dot P_K(1) = e \cdot E_{2K} \cdot V_K(2)/c\,, \\ \dot P_K(2) = e \cdot E_{2K}(1 - V_K(1)/c)\,, \\ \dot P_K(3) = 0\,, \end{cases} \tag{28}$$

where e is the particle electric charge and $V_K(i)$ are the particle velocity components. From eqs. (28) we get the parametric equations

$$X_K = (P_K(2) - P_{0K}(2))\left(\frac{\varepsilon^2}{a^2} - 1\right) A a 3/c + (P_K(2)^3 - P_{0K}(2)^3)\, cA + X_{0K}\,, \tag{29}$$

$$Y_K = (P_K(2)^2 - P_{0K}(2)^2)\, A + Y_{0K}\,, \tag{30}$$

$$Z_K = (P_K(2) - P_{0K}(2))\, P_K(3)\, A\, 6a + Z_{0K}\,, \tag{31}$$

$$t \;= (P_K(2)^3 - P_{0K}(2)^3)\, A + (P_K(2) - P_{0K}(2))\left(1 + \frac{\varepsilon^2}{a^2}\right) \Big/ 2eE_{2K}\,, \tag{32}$$

where

$$a = [Et_0 - cP_0(1)]\,, \qquad \varepsilon = [m_0^2 c^4 + P_0(3)^2 c^2]^{\frac12}\,, \qquad A = c^2/(6e \cdot E_{2K}\, a^2)\,,$$

Et_0 is the initial total energy of the particle.

From (32) we obtain

$$P_K(2)^3 + 3qP_K(2) - 2r(t) = 0 \tag{33}$$

resulting after the introduction of the parameters

$$q = (a^2 + \varepsilon^2)/c^2\,, \qquad r(t) = P_{0K}(2)^3/2 + P_{0K}(2)\, 3(a^2 + \varepsilon^2)/2c^2 + t \cdot c/2A\,.$$

Being $(q^3 + r^2) > 0$, the solution of eq. (33) is given by

$$P_K(2) = [r(t) + (q^3 + r(t)^2)^{\frac12}]^{\frac13} + [r(t) - (q^3 + r(t)^2)^{\frac12}]^{\frac13}\,, \tag{34}$$

which allows us to express the particle coordinates (eqs. (29)-(31)) as functions of time.

b) In the system K'' for $E_{2K} \neq B_{3K}$ the analytic solution is built up as follows.

The relativistic-motion equations of a particle in the system of coordinates K'' can be expressed in the form

$$\dot P''(1) = \Omega_0 P''(2)/\gamma\,, \tag{35}$$

$$\dot P''(2) = -\,\Omega_0 P''(1)/\gamma\,, \tag{36}$$

$$\dot P''(3) = e \cdot E_3''\,, \tag{37}$$

where $\Omega_0 = eB_3''/m_0 c$ and $\gamma(P'') = [1 + (P''/m_0 c)^2]^{\frac12}$.

We now define the parameters

$$\rho_{03} = P_3''(3)/m_0 \gamma_0 c\,, \qquad \gamma_0 = \gamma(P_0'')\,,$$

$$k = [1 - \rho_{03}^2]^{\frac12}\,,$$

$$\tau_0 = \rho_{03}/m_0 \gamma_0 ck\,,$$

$$\tau(t'') = (eE_3'' t'' + \rho_{03})/m_0 \gamma_0 ck\,,$$

$$\Phi(\tau) = \frac{B_3''}{E_3''}\,[\sinh^{-1}\tau - \sinh^{-1}\tau_0]\,,$$

where $\boldsymbol{P}_0''$ is the initial particle momentum.

The analytic solution of eqs. (35)-(37) is found to be given by

$$X''(t'') = \frac{1}{\Omega_0 m_0}\,[\boldsymbol{P}''_0(1)\sin\Phi - \boldsymbol{P}''_0(2)(\cos\Phi - 1)]\,, \tag{38}$$

$$Y''(t'') = \frac{1}{\Omega_0 m_0}\,[\boldsymbol{P}''_0(1)(\cos\Phi - 1) + \boldsymbol{P}''_0(2)\sin\Phi]\,, \tag{39}$$

$$Z''(t'') = \frac{m_0 c^2 \gamma_0}{eE''_3}\,[k\sqrt{1+\tau^2(t'')} - 1]\,, \tag{40}$$

where t'' is the proper time of the system $K''(t'' = t')$.

This solution has now to be transformed back in the laboratory frame L.

The transformation from K'' to K' is determined by a back-rotation of system K'' by the angle $-\theta$ about the X'' axis:

$$\left\{\begin{aligned} X' &= X''\,,\\ Y' &= Y''\cos\theta + Z''\sin\theta\,,\\ Z' &= -Y''\sin\theta + Z''\cos\theta\,. \end{aligned}\right. \tag{41}$$

The relativistic transformation from K' to K is given by

$$X_K(t'') = \gamma_K[X'(t'') + \beta_K t'' c]\,, \tag{42}$$

$$Y_K(t'') = Y'(t'')\,, \tag{43}$$

$$Z_K(t'') = Z'(t'')\,, \tag{44}$$

$$t_K(t'') = \gamma_K[t'' + \beta_K X_K(t'')/c]\,. \tag{45}$$

Equation (45) gives the proper time t_K of the system K (the same time of the system L).

The motion is defined as a time function by eqs. (42)-(44) according to eq. (45), where the time t_K is a function of time t''. The particle coordinates are finally translated to the system L by the transformations

$$\left\{\begin{aligned} X_L &= X_K C(1,1) + Y_K C(2,1) + Z_K C(3,1)\,,\\ Y_L &= X_K C(1,2) + Y_K C(2,2) + Z_K C(3,2)\,,\\ Z_L &= X_K C(1,3) + Y_K C(2,3) + Z_K C(3,3)\,. \end{aligned}\right. \tag{46}$$

5. – Numerical examples.

An electron orbit is constructed starting from its initial momentum in a spatial cell two millimeters wide, where the electric and magnetic fields can be assumed to be constant in time and uniform in space (fig. 12).

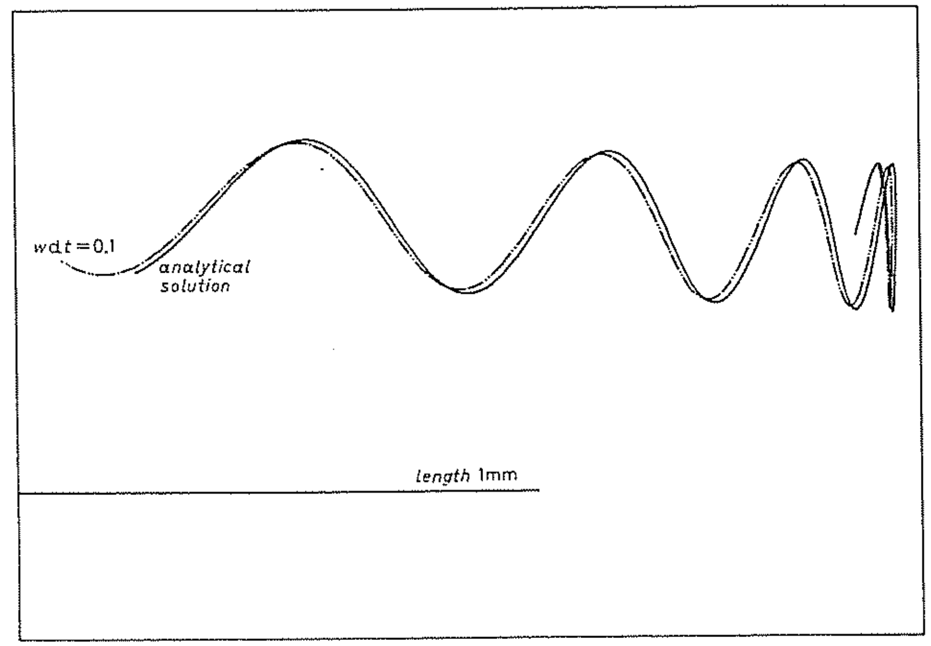

Fig. 12. – Projection on the (Z, X)-plane (of the system L) of the electron orbit calculated with the analytical solution (46) outlined in the present work. The motion parameters are the same of the orbit drawn in fig. 1.

The orbits obtained by the Boris mover tend to the ones constructed by means of the analytical solution with decreasing $\omega\,\delta t$ (see fig. 1).

On the contrary, the deviation from the analytic solution becomes very high when $\omega\,\delta t > 1$.

RIASSUNTO

Usando il metodo di Boris, oppure altri metodi standard, l'accumulazione degli errori nella determinazione numerica delle orbite relativistiche di particelle cariche in campi elettrici e magnetici diventa intollerabile, se il numero dei punti impiegati per costruire le orbite stesse è mantenuto basso. Una soluzione analitica diviene quindi utile, e questa costituisce l'argomento del presente lavoro.

Аналитическое рассмотрение релятивистского движения заряженных частиц в электрическом и магнитном полях.

Резюме (*). — В рамках стандартных методов погрешность при вычислении релятивистской орбиты заряженных частиц в электронном и магнитном полях становится чрезмерно большой, если для построения орбиты часицы используется ограниченное число точек. В связи с этим конструируется аналитическое решение.

(*) *Переведено редакцией.*