Exact and deterministic Hamiltonian description of wave-like features in classical and quantum physics

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ABSTRACT - The indeterministic character of physical laws is generally considered to be the most important consequence of quantum physics. A deterministic point of view, however, together with the possibility of well defined Hamiltonian trajectories, emerges as the most natural one from the analysis of the time-independent Helmholtz-like equations encountered both in Classical Electromagnetism and in Wave Mechanics. In the former case the rays of a beam are described, for any set of boundary conditions, by an exact approach avoiding the geometrical optics approximation; in the latter case the same pattern of lines is shown to provide the trajectories of a quantum particle beam, and to correspond to a set of deterministic dynamical laws containing the classical ones as a simple limiting case.

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1. Introduction

Due to the disdainful attitude of influent Founding Fathers such as Heisenberg and Einstein, the main *alternative* interpretation of Quantum Mechanics - the "hidden variables" point of view proposed by de Broglie **[1-3]** and Bohm **[4,5]** - did not enter in the mainstream of Physics, and was forced to develop into a separate, somewhat esoteric and almost heretical "church" **[6]**.

In the present (simple, but not necessarily simplistic) work we make use of an approach bearing some analogies with that alternative standpoint, but bypassing its uncomfortable Hamilton-Jacobi equations (requiring an often unattainable *generating function*) in favour of a set of easily integrable Hamiltonian equations. These equations provide an exact and deterministic description of the *quantum* motion of a particle beam, and contain the *classical* dynamical laws as a particular case, thus suggesting that the standard probabilistic treatment of quantum features may constitute the best approach when a detailed information is lacking, but does not necessarily reflect an intrinsically indetermined nature of physical reality.

As is well known (and as we shall see in the next Section), the **Helmholtz** equation, describing a wide family of classical monochromatic wave-like phenomena, may be reduced to a system of two coupled equations (eqs. (5) of the present paper). The *first* of these equations is usually *truncated*, by neglecting the term coupling it to the *second* equation. In such an *incomplete form* it provides, by itself alone, the set of "rays" which characterizes the so-called *geometrical optics approximation*. No further contribution to the ray geometry is given, in this limit, by the second of eqs. (5).

In the present paper the coupled equation system (5) is shown to lead, *without any omission or approximation*, to a Hamiltonian ray-tracing set of equations (our eqs. (13)), providing the *exact* description of a family of wave-like phenomena much wider than that allowed by the standard geometrical optics and including, for instance, wave diffraction and interference. The term $\frac{\nabla^2 R}{R}$, usually dropped from the *first* of eqs. (5) in the context of standard geometrical optics, is taken into account and shown to be of crucial importance: the rays of a beam turn out to be mutually correlated, indeed, by its gradient, acting *perpendicularly* to the rays themselves and determining therefore their geometry without altering the amplitude of their velocity; and an equally crucial importance is shown to be attached to the *second* of eqs. (5).

The *exact* ray-tracing system (13) deduced from the **Helmholtz** equation is then shown to be strictly analogous to a *novel*, *exact* and *deterministic* dynamical Hamiltonian system (our eqs.(27)), deduced from the *time-independent* **Schrödinger** equation (which is itself a Helmholtz-like equation), and providing the trajectories and the motion law of a quantum particle beam. The term neglected in the context of standard geometrical optics is shown, in its turn, to coincide (aside from constant factors) with the *quantum potential* described by Bohm and de Broglie.

For the first time, therefore, in the 55 years elapsed from Bohm's works, the basic character of the "force" deduced from this *quantum potential* (i.e. the fundamental property of being transversal with respect to the particle velocities, thus *preserving their amplitude*) is discovered, stressed and exploited.

A unique Hamiltonian system (our eqs.(28)), to which both eqs. (13) and (27) may be reduced, is then obtained for an arbitrary beam consisting, indifferently, either of *classical* electromagnetic rays or of *quantum* particles. Such a system is numerically solved here in the simple 2-dimensional case (30) for a beam passing through a single slit, and the *diffractive* behaviour of its solutions is clearly evidenced.

We point out again that the main intuition leading to these novel results stems from the formal analogy between the *time independent* **Schrödinger** equation and the **Helmholtz** equation: an analogy, indeed, which allows to go beyond *classical dynamics* by means of the same mathematics allowing to go beyond the standard *geometrical optics approximation*. A direct proof that a deterministic, classical-looking description is possible not only

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in principle, but in easy practice, is provided, therefore, in a simple and natural way, avoiding the conceptually useless complication - as far as a basic principle has to be established - involved by the *time dependent* Schrödinger equation: an apparent generality whose result is only to hinder the demonstration that particle trajectories *do exist*.

2. Helmholtz equation and geometrical optics

In order to establish the mathematical formalism to be extended, later on, to the quantum treatment of a particle beam, let us start from a *classical* case of wave-like behaviour.

Although many kinds of waves would lend themselves to the considerations we have in mind here, we shall refer, in order to fix ideas, to a monochromatic electromagnetic wave beam, with a time dependence $\div \exp(i\omega t)$, travelling through an isotropic and inhomogeneous dielectric medium. Its basic features are accounted for by the Helmholtz equation

$$\nabla^2\psi + (n\,k_0)^2\psi = 0 \quad , \qquad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; ψ represents any component of the electric or magnetic field; $n(x,y,z)$ is the refractive index of the medium, and

$$k_0 \equiv \frac{2\pi}{\lambda_0} = \frac{\omega}{c} \quad , \qquad (2)$$

with obvious meaning of λ_0 and c. The phase velocity is given, in its turn, by

$$v_{ph}(x,y,z) = c/n(x,y,z) \,. \qquad (3)$$

Because of its time-independence, eq.(1) doesn't directly describe, of course, any propagation phenomenon: it only determines, together with the boundary conditions, the fixed space frame where propagation occurs.
By performing the quite general replacement

$$\psi(x,y,z) = R(x,y,z)\,e^{\,i\varphi(x,y,z)} \quad , \qquad (4)$$

with real $R(x,y,z)$ and $\varphi(x,y,z)$, and separating the real from the imaginary part, eq.(1) splits into the well known **[7]** and strictly equivalent system of coupled equations

$$\begin{cases} (\underline{\nabla}\,\varphi)^2 - (nk_0)^2 = \dfrac{\nabla^2 R}{R} \\ \underline{\nabla}\cdot(R^2\,\underline{\nabla}\,\varphi) = 0 \end{cases} \qquad (5)$$

where $\underline{\nabla} \equiv \partial/\partial\boldsymbol{r} \equiv (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, and the second of eqs.(5) expresses the constancy of the flux of the vector $R^2\underline{\nabla}\varphi$ along any tube formed by the lines of $\underline{\nabla}\varphi$ itself, i.e. normally to the phase surfaces $\varphi(x,y,z) = const.$
When the space variation length, L, of the amplitude $R(x,y,z)$ may be assumed to satisfy the condition $k_0 L >> 1$, the first of eqs.(5) is well approximated by the *eikonal equation*

$$(\underline{\nabla}\,\varphi)^2 \cong (nk_0)^2 \quad , \qquad (6)$$

decoupled from the second of eqs.(5) (whose presence is generally neglected) and allowing the so-called *geometrical optics approximation*, which describes the wave propagation in terms of "rays" travelling along the field lines of the *wave vector*

$$\boldsymbol{k} = \underline{\nabla}\,\varphi \qquad (7)$$

independently from the amplitude distribution $R(x,y,z)$ of the beam. To be sure, by multiplying eq.(6), for convenience, by the constant factor $\frac{c}{2\,k_0}$, we obtain the relation

$$D(\boldsymbol{r},\boldsymbol{k}) \equiv \frac{c}{2\,k_0}[\,k^2 - (n\,k_0)^2\,] \cong 0 \,, \qquad (8)$$

(where $\boldsymbol{r} \equiv (x,y,z)$), whose differentiation

$$\frac{\partial D}{\partial \boldsymbol{r}}\cdot d\,\boldsymbol{r} + \frac{\partial D}{\partial \boldsymbol{k}}\cdot d\,\boldsymbol{k} = 0 \qquad (9)$$

directly provides, for any assigned refractive function $n(\boldsymbol{r})$, both the geometrical form of the rays and their motion law in the simple Hamiltonian form

$$\begin{cases} \dfrac{d\,\boldsymbol{r}}{d\,t} = \dfrac{\partial D}{\partial \boldsymbol{k}} = \dfrac{c\,\boldsymbol{k}}{k_0} \\ \dfrac{d\,\boldsymbol{k}}{d\,t} = -\dfrac{\partial D}{\partial \boldsymbol{r}} = \dfrac{c}{2\,k_0}\dfrac{\partial}{\partial \boldsymbol{r}}(nk_0)^2 \end{cases} \qquad (10)$$

where a ray velocity $\boldsymbol{v}_{ray} = \frac{c\,\boldsymbol{k}}{k_0}$ is implicitly defined. We may observe that $v_{ray} \equiv |\boldsymbol{v}_{ray}| = c$ when $k = k_0$, and that $v_{ray}\,v_{ph} = c^2$.
We conclude the present Section by recalling Fermat's variational principle, according to which any optical ray travelling between two points A,B shall follow a trajectory satisfying the condition

$$\delta \int_A^B k\,ds = 0 \,, \qquad (11)$$

where $k = |\,\boldsymbol{k}\,|$ and ds is an element of a (virtual) line connecting A and B.

3. Beyond the geometrical optics approximation

Let us consider now the first of eqs.(5) in its *complete* form, arriving therefore at the *exact* relation, generalizing the function $D(\boldsymbol{r},\boldsymbol{k})$ of eq.(8),

$$D(\boldsymbol{r},\boldsymbol{k}) \equiv \frac{c}{2\,k_0}[\,k^2 - (n\,k_0)^2 - \frac{\nabla^2 R}{R}\,] = 0 \,, \qquad (12)$$

whose differentiation, formally coinciding with eq. (9), leads to the *exact* Hamiltonian ray-tracing system

$$\begin{cases} \dfrac{d\,\boldsymbol{r}}{d\,t} = \dfrac{\partial D}{\partial \boldsymbol{k}} = \dfrac{c\,\boldsymbol{k}}{k_0} \\ \dfrac{d\,\boldsymbol{k}}{d\,t} = -\dfrac{\partial D}{\partial \boldsymbol{r}} = \dfrac{c}{2\,k_0}\dfrac{\partial}{\partial \boldsymbol{r}}[(nk_0)^2 + \dfrac{\nabla^2 R}{R}] \end{cases} \qquad (13)$$

The system (13) completely avoids the standard approximation of geometrical optics, although fully retaining the idea of electromagnetic "rays" travelling along the field lines of $\boldsymbol{k} \equiv \underline{\nabla}\varphi$, which depend, however, on the wave amplitude distribution *R(x,y,z)* of the beam. In order to exploit this dependence we must recall the presence of the *second* of eqs. (5), which may be written in the form

$$\underline{\nabla} \cdot (R^2 \underline{\nabla}\varphi) \equiv 2R\, \underline{\nabla}R \cdot \underline{\nabla}\varphi + R^2 \underline{\nabla} \cdot \underline{\nabla}\varphi = 0 \qquad (14)$$

Since no new ray trajectory may suddenly arise in the space region spanned by the beam, we must have, of course, $\underline{\nabla} \cdot \underline{\nabla}\varphi = 0$, so that eq.(14) splits into the system

$$\begin{cases} \underline{\nabla} \cdot \underline{\nabla}\varphi = 0 \\ \underline{\nabla}R \cdot \underline{\nabla}\varphi = 0 \end{cases} \qquad (15)$$

where the second equation is automatically entailed by the first one. The values of the function *R(x,y,z)* are therefore constant (i.e. "transported") along the field lines of $\boldsymbol{k} \equiv \underline{\nabla}\,\varphi$, to which $\underline{\nabla}\, R$ turns out to be *perpendicular*, and this transverse character is shared by the gradient $\frac{\partial}{\partial \boldsymbol{r}} \frac{\nabla^2 R}{R}$. The amplitude v_{ray} of the ray velocity shall remain, in vacuum, equal to c all along its trajectory, because such a gradient may only modify the *direction*, but not the *absolute value*, of the wave vector $\boldsymbol{k}$: the only possible changes of $|\boldsymbol{k}|$ could be due, in a medium different from vacuum, to its refractive function *n(x,y,z)*.

Thanks to its constancy along each ray trajectory the function *R(x,y,z)* once assigned on the launching surface from where the ray beam is assumed to start, may be numerically built up step by step, together with its derivatives, in the whole region crossed by the beam. As we shall see in Sect.7, indeed, the *exact* equation system (13) lends itself to an easy numerical solution, even in physical cases where the standard geometrical optics *approximation* is completely inapplicable.

4. The time-independent Schrödinger equation

The *classical* motion of a mono-energetic beam of non-interacting particles of mass m through a force field deriving from a potential energy *V(x,y,z)* not explicitly depending on time may be described for each particle of the beam, as is well known, by means of the so-called "reduced" (or "time-independent") Hamilton-Jacobi equation **[7]**

$$(\underline{\nabla}\, S)^2 = 2\, m\, (E - V)\ , \qquad (16)$$

where E is the total energy, and one of the main properties of the function *S(x,y,z)* is that the particle momentum is given by

$$\boldsymbol{p} = \underline{\nabla}\, S\ . \qquad (17)$$

Recalling Maupertuis' variational principle

$$\delta \int_A^B p\, ds \equiv 0\ , \qquad (18)$$

with $p = |\,\boldsymbol{p}\,|$, the formal analogy between eqs.(6,7,11) on one side, and eqs.(16-18) on the other side, suggests, as is well known, that the *classical* particle trajectories could constitute the *geometrical optics approximation* of an equation (analogous to the Helmholtz eq.(1)), which is immediately obtained by means of the substitutions

$$\begin{cases} \varphi = \dfrac{S}{a} \quad \text{and therefore} \\ \boldsymbol{k} = \underline{\nabla}\varphi = \dfrac{\underline{\nabla} S}{a} = \dfrac{\boldsymbol{p}}{a}\,; \\ k_0 \equiv \dfrac{2\pi}{\lambda_0} = \dfrac{\sqrt{2mE}}{a} \equiv \dfrac{p_0}{a} \\ n^2(x,y,z) = 1 - \dfrac{V(x,y,z)}{E} \end{cases} \qquad (19)$$

where the parameter a represents a constant *action* whose value is *a priori* arbitrary - as far as the relations (19) are concerned - but is imposed by the history itself of Quantum Mechanics :

$$a = \hbar \cong 1.0546 \times 10^{-27}\, erg \cdot s\ . \qquad (20)$$

The equation obtained from the Helmholtz equation (1) by means of the substitutions (19) and (20) takes up the form

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\,\psi = 0\ , \qquad (21)$$

which is the standard *time-independent* Schrödinger equation.
By applying now to eq.(21) the same procedure leading from eq.(1) to eqs.(5), and assuming therefore

$$\psi(x,y,z) = R(x,y,z)\, e^{\,i\, S(x,y,z)/\hbar} \qquad (22)$$

eq.(21) splits **[8]** into the coupled system

$$\begin{cases} (\underline{\nabla}\, S)^2 - 2m(E - V) = \hbar^2 \dfrac{\nabla^2 R}{R} \\ \underline{\nabla} \cdot (R^2\, \underline{\nabla}\, S) = 0 \end{cases} \qquad (23)$$

By taking the gradient of the first of eqs.(23) we get moreover

$$\left(\frac{\underline{\nabla}\, S}{m} \cdot \underline{\nabla}\right) \frac{\underline{\nabla}\, S}{m} + \frac{\underline{\nabla}\, V}{m} = \frac{\hbar^2}{2\, m^2} \underline{\nabla}\left(\frac{\nabla^2 R}{R}\right). \qquad (24)$$

Eq.(24), together with the second of eqs.(23), is often interpreted as describing, in the "classical limit" $\hbar \to 0$ (whatever such a limit may mean), a "fluid" of particles with mass m and velocity $\frac{\underline{\nabla}\, S}{m}$, moving in an external potential *V(x,y,z):* an interpretation consistent with the probabilistic character usually ascribed to the Schrödinger equation.

5. Hamiltonian description of quantum particle motion

Let us now observe that, by simply maintaining eq.(17), the first of eqs.(23) may be written in the form of a generalized, time-independent Hamiltonian

$$H\,(\boldsymbol{r},\boldsymbol{p}\,) \equiv \frac{p^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = E\ , \qquad (25)$$

including the "new" and crucial term $\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$, to be commented later on.
By differentiating eq.(25) we get the relation

$$\frac{\partial H}{\partial \boldsymbol{r}}\cdot d\,\boldsymbol{r} + \frac{\partial H}{\partial \boldsymbol{p}}\cdot d\,\boldsymbol{p} = 0 \qquad (26)$$

leading to a Hamiltonian dynamical system of the form

$$\begin{cases} \frac{d\,\boldsymbol{r}}{d\,t} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p}}{m} \\ \frac{d\,\boldsymbol{p}}{d\,t} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\frac{\partial}{\partial \boldsymbol{r}}[V(\boldsymbol{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}] \end{cases} \qquad (27)$$

strictly similar to the ray-tracing system (13). If we envisage the system (27) for what it appears to be, without superimposing any interpretative prejudice, it is quite evident that its mathematical treatment is the same employed in the classical ray-tracing case, including the fact that the function *R(x,y,z)* is "transported" along the field lines of $\boldsymbol{p} \equiv \underline{\nabla}\, S$, to which $\underline{\nabla}\, R$ turns out to be perpendicular. The gradient $\frac{\partial}{\partial \boldsymbol{r}}\frac{\nabla^2 R}{R}$, in its turn, remains tangent to the wave-front, without acting on the *amplitude* of the particle velocity (but modifying, in general, its *direction*). The only possible amplitude changes could be due to the presence of an external potential *V(x,y,z)*.
Once more, thanks to its constancy along each trajectory, the function *R(x,y,z)* may be assigned on the launching surface from where the beam is assumed to start, and numerically built up step by step, together with its derivatives, in the whole region spanned by the motion of the beam.

6. The unique dimensionless Hamiltonian system

A quite expedient step is now the passage to the new, dimensionless variables $\underline{\xi}, \underline{\rho}, \tau$ defined as the ratio between, ***r***, ***p*** and *t*, respectively, and $\lambda_0 \equiv 2\pi\,\hbar / p_0$ for the space variables, $p_0 \equiv (2mE)^{1/2}$ for the momentum variables (so that $\rho_0 = 1$), and $\frac{\lambda_0}{p_0 / m}$ for the time variable.
The equation system (27) takes up therefore the form

$$\begin{cases} \frac{d\,\underline{\xi}}{d\,t} = \underline{\rho} \\ \frac{d\,\underline{\rho}}{d\,t} = -\frac{\partial}{\partial\,\underline{\xi}}[\frac{V(\underline{\xi})}{2E} - \frac{1}{8\,\pi^2}\;G(\underline{\xi})] \end{cases} \qquad (28)$$

with

$$G(\underline{\xi}) = \frac{1}{R}(\frac{\partial^2 R}{\partial\,\xi^2} + \frac{\partial^2 R}{\partial\,\eta^2} + \frac{\partial^2 R}{\partial\,\zeta^2}); \qquad \underline{\xi} \equiv (\xi,\eta,\zeta) \equiv (x/\lambda_0,\; y/\lambda_0,\; z/\lambda_0) \qquad (29)$$

It may be observed that no direct reference is present, in the dimensionless form (28) assumed by the *quantum* dynamical system (27), to the mass of the moving particles, and not even to $\hbar$.
Let us also observe that *the same dimensionless form (28)* is taken up by the ray-tracing system (13) - relevant to the *classical* electromagnetic case - by simply assuming $\tau = \frac{c\,t}{\lambda_0}$ and replacing $\underline{\rho}$ with $\frac{\boldsymbol{k}}{k_o} \equiv \frac{\boldsymbol{v}_{ray}}{c}$ and $\frac{V(x,y,z)}{E}$ with *[1- n^2(x,y,z)]*, in agreement with the relations (19).

Once assigned on the launching surface of the beam, the function *G(*$\underline{\xi}$*)* may be numerically determined step by step, in principle, together with its derivatives, by means of an interpolation process iterated along the full set of trajectories of the beam and connecting each step to the previous ones. This function, due to the wave amplitude distribution of the beam on the advancing wave-front, turns out to be the same - in correspondence with the same boundary conditions - for *classical* electromagnetic rays as well as for *quantum* material particles, although it has obviously nothing to do, in the electromagnetic case, with quantum features. In its absence, however, the system (28) would simply describe the *classical* motion of each particle of the beam. Due to the small coefficient $\frac{1}{8\pi^2}$, the transverse gradient $\frac{\partial\,G}{\partial\,\underline{\xi}}$ acts along the trajectory pattern in a soft and cumulative way: a fact granting the main justification for omitting such a term, as is done both in classical dynamics and in the standard geometrical optics approximation.

The trajectory pattern, in its turn, is a stationary structure determined at the very outset in a way somewhat reminding the spirit of classical variational principles, such as the ones of Fermat and Maupertuis. For any set of boundary conditions imposed to the function *R(x,y,z)* on the launching surface of the beam, and for any assigned refractive medium (or force field), the system (28) provides both a "weft" of "rails" and a motion law to which particles (or rays) are deterministically bound, *showing no trace of probabilistic features*.

The modern point of view of Quantum Mechanics on indeterminism has nothing to do, as is well known, with the naive idea of a disturbance due to the observer, which would imply an underlying *deterministic* situation "blurred" by the observation device. Indeterminism is currently conceived, in fact, as an intrinsic natural property, forbidding, even in principle, to assign a definite trajectory to a moving particle, and reserving to the observer the subtle role of inducing (in general) the collapse of the observed system from a superposition of its possible states into a single one of them, according to well defined probabilities.

Contrary to this point of view, however, each particle (as well as each electromagnetic ray) of the beam turns out to be conceivable, on the basis of the present analysis, as *starting and remaining* on a well definite trajectory. Such a trajectory belongs to a pattern which is *a priori* fixed, as a whole, by the properties of the medium and by the values assigned to the beam amplitude distribution *R(x,y,z)* on the launching surface.

The system (28) provides, in conclusion, a set of dynamical laws which replace - and contain as a limiting case, when the transverse gradient $\frac{\partial}{\partial \boldsymbol{r}}\frac{\nabla^2 R}{R}$ may be assumed to be negligible - the classical ones. Let us observe that the possibility of neglecting such a term, and of obtaining therefore a classical-looking description, may turn out to be limited to a simple portion (typically, the central part) of a beam. In striking divergence from the classical dynamical laws, however, the new set of equations, because of its equivalence with a Helmholtz-like equation, requires in general the full set of boundary conditions for the determination of each trajectory of the beam.

7. Wave-like features in Hamiltonian form

Although an accurate and general numerical treatment lies beyond the aims of the present paper, we propose here the application of the equation system (28) to the propagation of a collimated beam injected at $\zeta = 0$ parallel to the ζ-axis, and centered at $\xi = 0$, in order to simulate wave diffraction through a single slit.

The problem may be faced by taking into account for simplicity sake (but with no substantial loss of generality) either a (quantum) *particle beam* in the absence of external fields ($V = 0$), or a (classical) *electromagnetic beam* in vacuum ($n^2 = 1$), with a geometry allowing to limit the computation to the trajectories lying on the (ξ,ζ)-plane. The Hamiltonian system (28) takes up therefore the form

$$\begin{cases} \dfrac{d\xi}{d\tau} = \rho_x \\ \dfrac{d\zeta}{d\tau} = \rho_z \\ \dfrac{d\rho_x}{d\tau} = \dfrac{1}{8\pi^2}\dfrac{\partial}{\partial\xi}G(\xi,\zeta) \\ \dfrac{d\rho_z}{d\tau} = \dfrac{1}{8\pi^2}\dfrac{\partial}{\partial\zeta}G(\xi,\zeta) \end{cases} \qquad (30)$$

with

$$G(\xi,\zeta) = \frac{1}{R}\left(\frac{\partial^2 R}{\partial\xi^2} + \frac{\partial^2 R}{\partial\zeta^2}\right); \qquad (31)$$
$$\rho_x(\zeta=0)=0;\ \rho_z(\zeta=0)=\rho_0=1$$

and a suitable amplitude distribution $R(\xi,\zeta=0)$ (from whose normalization the function G is obviously independent) imposed at $\zeta = 0$.

Because of the transverse nature of the gradient of $G(\xi,\zeta)$, the *amplitude* of the vector $\underline{\rho}$ remains unchanged (in the absence of external fields and/or refractive effects) along each trajectory, leading therefore to the relation

$$\rho_z = \sqrt{\rho_o^{\,2} - \rho_x^{\,2}} \equiv \sqrt{1-\rho_x^{\,2}}\,, \qquad (32)$$

which may advantageously replace the fourth equation of the Hamiltonian system (30). Two possible models of the amplitude distribution $R(\xi,\zeta=0)$ are obtained by assuming

- a Gaussian distribution centered at $\xi = 0$, in the form

$$R_1(\xi;\zeta=0) \div e^{-\left(\frac{x}{w_0}\right)^2} \equiv e^{-\varepsilon^2\xi^2} \qquad (33)$$

(with constant w_0 and $\varepsilon = \frac{\lambda_0}{w_0} \le 1$), a functional form suggested by its smooth analytical behaviour; or

- an algebraic distribution, in the form

$$R_2(\xi;\zeta=0) \div \frac{1}{1+\left(\frac{x}{w_0}\right)^{2N}} \equiv \frac{1}{1+(\varepsilon\xi)^{2N}} \qquad (34)$$

(with integer N), which allows to represent even a quite flat central region, widening with increasing N. We show in **Fig.1** both the distributions R_1 and R_2, with $\varepsilon = 0.1$ and $N = 1$, and in **Fig.2** the corresponding functions

$$G_{1,2}(\xi;\zeta=0) = \frac{1}{R_{1,2}}\frac{d^2R_{1,2}}{d\xi^2} \qquad (35)$$

determining the launching conditions at $\zeta=0$. It is seen that rather similar distributions $R_{1,2}$ may lead to quite different $G_{1,2}$ *and therefore to quite different trajectory patterns.* In our preliminary computations the functions $G_{1,2}(\xi;\zeta>0)$ are built up step by step by means of a 3-points Lagrange interpolation. As predicted by the standard diffraction theory **[9]**, no "fringe" is found in the Gaussian case of **Fig.3** (due to the fact that the Fourier transform of the distribution R_1 consists of another Gaussian function), while "fringes" appear (in the form of gathering trajectories) in **Fig.4** for the algebraic initial distribution R_2, focusing closer to the launching plane for higher values of ε. We shall not discuss here the specific form of these fringes, since the basic result to be pointed out is their very appearance in the context of our Hamiltonian approach.

No further difficulty would be encountered in the case of *two* beams, injected parallel to the ζ-axis at $\zeta = 0$ and centered, on the ξ-axis, at two symmetrical points $\xi = \pm\,\xi_0$, in order to simulate both their diffraction and their interference through a double slit.

8. Discussion and conclusions

A certain analogy may be observed between the results of the present work and the ones previously published by one of the Authors (A.O.) in a quite different context **[10-12]**. Another obvious analogy is found with Refs.**[13,14]** (based on Bohm's approach) which are hindered, however, by a Hamilton-Jacobi set of equations which would need, in general, an often unattainable *generating function*: an obstacle which is avoided by an entangled solution method requiring the previous knowledge of the wave function ψ. Such a harsh method (and logical discontinuity) should be compared with our directly integrable set of Hamiltonian motion laws. To be sure, the mathematical complexity of the set of particle trajectories presented (in the *interference* case) in Refs.**[13,14]** is so great that these lines are simply reproduced (without any improvement) in Refs.**[6a,b]**. Let us stress, incidentally, that we analyze here, for the first time, the *diffractive* case, showing the dependence both of the ray geometry and of the motion law on the launching beam amplitude distribution ***R(r)*** (our eqs.(33) and (34)).

Our opinion is that Bohm did not convince the scientific community because he did not notice the implications, holding *even beyond the quantum case*, of the *time-independent* (and therefore Helmholtz-like) Schrödinger equation, allowing a simple, direct and natural connection with geometrical optics. While, in particular, the term $\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$ of eq. (25) has the dimensions and the behaviour of a potential field, exerting a real (transverse) force on the *quantum* particle beam, the corresponding term in eq.(12), concerning a *classical* electromagnetic ray beam, has an obviously different nature, but leads to a strictly similar "weft" of trajectories. A proper analysis of our Hamiltonian system in its general form (28) reveals indeed that the deviations of a particle beam from *classical dynamics* (or of a ray beam from *standard geometrical optics*) are entirely due, in any case, to the role of the gradient $\frac{\partial G}{\partial \underline{\xi}}$, arising from the beam distribution on the advancing wave-front and tangent to such a surface (thus affecting the beam geometry, but not its velocities).

A further, basic point to be stressed here is the influence of the full set of boundary conditions on the *form of the trajectories*, and on the *motion* along them, of each particle (or ray) of the beam: a point which concerns, however, *Wave Mechanics* as well as *Classical Electromagnetism* (together with whatever phenomenon may be described in terms of Helmholtz-like equations). Any attempt, indeed, to apply the time independent *Schrödinger* equation to a single particle (not belonging to a beam) would not appear to be more plausible than the application of the *Helmholtz* equation to a single ray.

We may conclude the present work by suggesting that, contrary to a well established opinion, a probabilistic description of the behaviour of a quantum particle beam, although representing a convenient approximation when a fully detailed information is lacking or unnecessary, doesn't inevitably supply the most exact possible approach. As we have shown, in fact, it is a straightforward task, *starting from the time-independent Schrödinger equation* and avoiding its usual indeterministic interpretation without encountering any logical contradiction, to obtain a deterministic description, where the particle trajectories maintain a classical-looking reality.

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FIGURE CAPTIONS

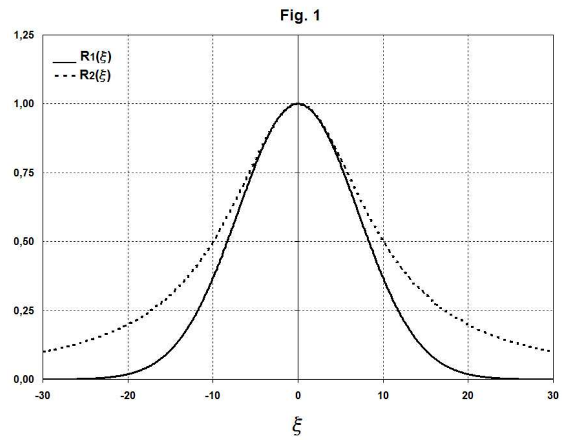

Fig.1 - Plot of the amplitude distributions $R_{1,2}$ assigned to the beam on the launching plane $\zeta = 0$, for $\varepsilon = \frac{\lambda_0}{w_0} = 0.1$,

- in the Gaussian case of eq.(33) (continuous line);
- in the algebraic case of eq.(34), with $N=1$ (dotted line).

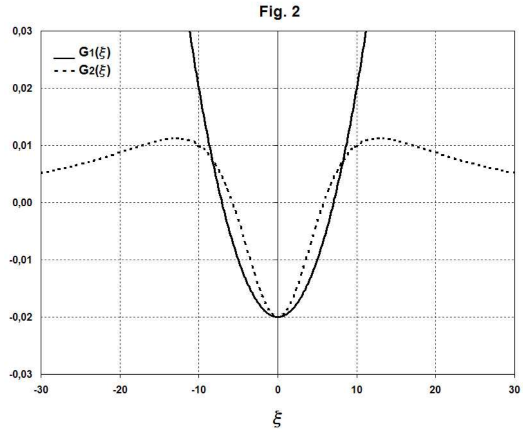

Fig.2 - Plot of the initial functions $G_{1,2}$ of eq. (35) corresponding to the distributions $R_{1,2}$ of FIG.1.

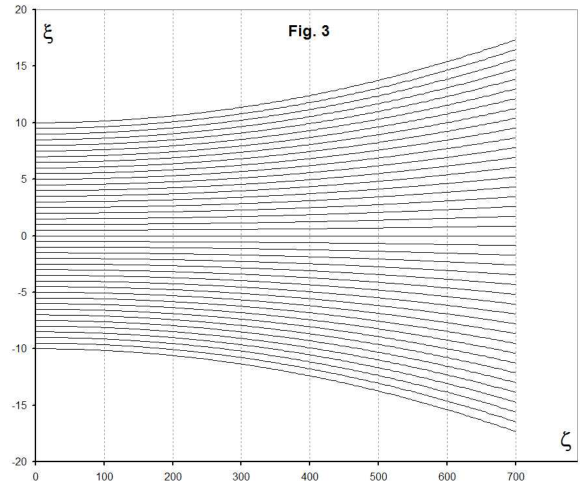

Fig.3 - Trajectory pattern on the (ξ,ζ)-plane, in the Gaussian case of FIG.1. The beam is truncated at $\zeta = 700$, in order to limit it to its most interesting part.

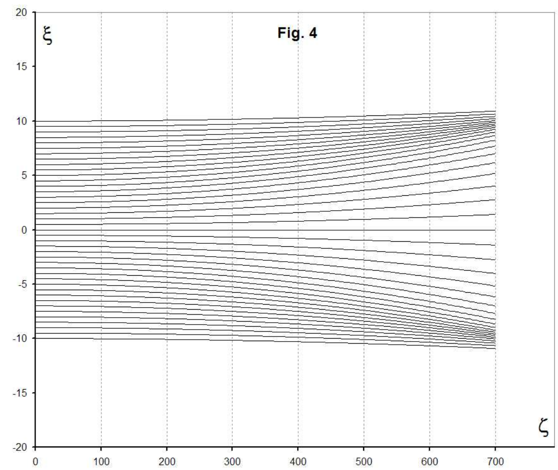

Fig.4 - Trajectory pattern on the (ξ,ζ)-plane, in the algebraic case of FIG.1.